CE 205: Finite Element Method: Homework III

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You must submit your source code for all coding problems. Show all work clearly. Plots must be labeled legibly and completely. This homework has THREE pages.

1. Make a plot of the three quadratic Lagrange interpolation functions

$$N_1(x) = \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)} \quad N_2(x) = \frac{(x_3 - x)(x_1 - x)}{(x_3 - x_2)(x_1 - x_2)} \quad N_3(x) = \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}$$

Show that these functions satisfy the partition of unity property. Can an element with displacement given by $u^{el}(x) = \sum_{i=1}^{3} N_i(x) u_i$ exactly represent a state of uniform strain? Show why or why not.

2. Consider the beam interpolation functions $N_i(x)$ and their derivatives $N_i'(x)$. Make a plot of these functions over $[0, L]$.

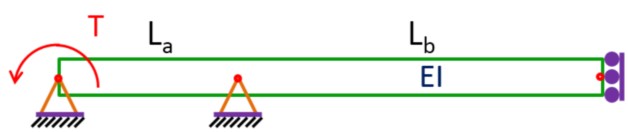

Figure 1: Two-span beam problem

3. Consider the two-span beam shown in Fig. 1 and do the following:

   (a) Write down the reduced stiffness matrix for this problem.

   (b) For the case $L_b = 2L_a = 2L$, find the reactions at the supports. Find and plot the deflections, slopes, and the maximum sectional stresses in the spans over $x = [0, 3L]$. You can assume a rectangular section $b \times h$.

   (c) Imagine that the rightmost roller support is replaced by a fixed support. Repeat the analyses in parts (a) and (b) above with these new boundary conditions.

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4. Consider an elastic beam of length L with uniform sectional properties EI. It is subjected to transverse point loads $+P$ at $x = L/4$, $-P$ at $x = 3L/4$, as well as a distributed transverse load per unit length $q(x) = q_0 \sin(2\pi x/L)$ along its entire length. Treating the beam as a single (beam) element, find the vector of consistent nodal loads, and make a sketch of these loads. Assume that $P > 0$ and $q_0 < 0$.
   Also show that these loads are statically equivalent to the original loads.
   Be careful about the sign and sense of the applied loads.

5. Let $\mathcal{U}$ be the elastic energy of a structural finite element with degrees of freedom vector $\{d\}$. Show that the (i, j) entry of the element stiffness matrix $[k]$ is given by

$$k_{ij} = \frac{\partial^2 \mathcal{U}}{\partial d_i \partial d_j}$$

6. Let $[\boldsymbol{\beta}]$ be the sub-matrix

$$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin(\phi) & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Show that the 6×6 transformation matrix $[\boldsymbol{\beta}^*]$ defined as

$$[\boldsymbol{\beta}^*] = \left[\begin{array}{c|c} [\boldsymbol{\beta}] & [\mathbf{0}] \\ \hline [\mathbf{0}] & [\boldsymbol{\beta}] \end{array}\right]$$

   is proper orthogonal. Note that $[\mathbf{0}]$ is the 3×3 zero matrix.

7. *Coding exercise* Consider the two-element plane frame example problem discussed in class. This is reproduced for your convenience in Fig. 2 below. Assume the frame elements have identical properties EI, EA, and L.

   (a) Write down the vector of unknown d.o.f and re-derive the $[K^*]$ matrix. Make sure to keep the truss and beam contributions separate.

   (b) Solve for all the unknown d.o.f using the MATLAB symbolic math toolbox. You will need to use it carefully to write down the results in a way that is human readable and insightful. One way to do so is to introduce the ratio ζ of flexural to axial stiffness

$$\zeta = \frac{EI/L^3}{EA/L}$$

   to represent the degrees of freedom. For instance, using ζ one has

$$u_3 = \frac{PL^3}{EI}\,\frac{36\,\zeta^2 + 60\,\zeta + 7}{12\,[4 + 3\,\zeta]}$$

(c) Find expressions for all the reaction forces and moments.

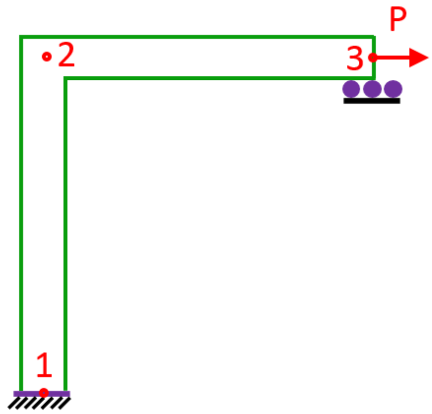

Figure 2: Right-angled two-member plane frame

8. Consider the right-angled plane frame shown in Fig. 3 below. Assume the frame elements have identical properties EI, EA, and L. Impose structural symmetry to solve the problem.

   (a) First, write down the vector of unknown d.o.f.s before and after imposing symmetry.

   (b) Find the $[K^*]$ matrix for the system after imposing symmetry.

   (c) Find expressions for all the unknown d.o.fs.

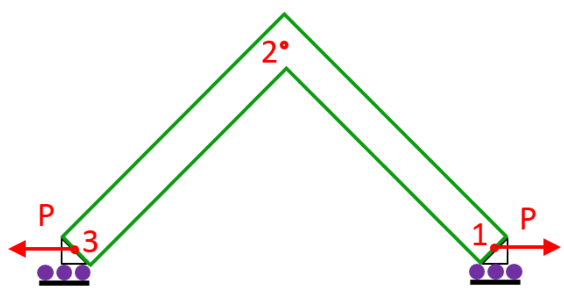

Figure 3: Inclined right-angled plane frame

9. Consider a variant of the previous problem in which the structure is still symmetric, but the included angle between the two members is not $\pi/2$, but some angle 2ω, where $\pi/4 \leq \omega \leq \pi/2$. Write down the $[K^*]$ after imposing symmetry in this case. Check that your $[K^*]$ simplifies as expected in the special cases $\omega = \pi/4$ and $\omega = \pi/2$.
